ALGEBRAIC INDEPENDENCE OF GENERALIZED MMM-CLASSES

JOHANNES EBERT

Abstract. The generalized Morita-Miller-Mumford classes of a smooth oriented manifold bundle are defined as the image of the characteristic classes of the vertical tangent bundle under the Gysin homomorphism. We show that if the dimension of the manifold is even, then all MMM-classes in rational cohomology are nonzero for some bundle. In odd dimensions, this is also true with one exception: the MMM-class associated with the Hirzebruch $\mathcal{L}$-class is always zero. We also show a similar result for holomorphic fibre bundles.

Contents

1. Introduction and statement of results

Let M be a closed oriented n-dimensional smooth manifold and let $\mathrm{Diff}^+(M)$ be the topological group of all orientation-preserving diffeomorphisms of M, endowed with the Whitney C^∞-topology. A *smooth oriented M-bundle* is a fibre bundle with structural group $\mathrm{Diff}^+(M)$ and fibre M. Let $Q \to B$ be a $\mathrm{Diff}^+(M)$-principal bundle. The *vertical tangent bundle* of the smooth oriented M-bundle $f : E := Q \times_{\mathrm{Diff}^+(M)} M \to B$ is the oriented n-dimensional vector bundle $T^f = T_v E := Q \times_{\mathrm{Diff}^+(M)} TM \to E$. A *smooth oriented closed fibre bundle of dimension n* is a map $f : E \to B$ such that for any component $C \subset B$, $f : f^{-1}(C) \to C$ is a smooth oriented M-bundle for some closed oriented n-manifold M. We sometimes abbreviate this term to *smooth fibre bundle*, because all manifold bundles we consider are oriented and have closed fibres.

If $f : E \to B$ is a smooth oriented fibre bundle, then the *Gysin homomorphism* (or umkehr homomorphism) $f_! : H^*(E) \to H^{*-n}(B)$ is defined (all cohomology groups in

this paper have rational coefficients, unless we explicitly state the contrary). Define a linear map

$$\kappa_E : H^*(BSO(n);\mathbb{Q}) \to H^{*-n}(B;\mathbb{Q}) \tag{1.1}$$

by

$$\kappa_E(c) := f_!(c(T_vE)) \in H^{k-n}(B);\ c \in H^*(BSO(n);\mathbb{Q}). \tag{1.2}$$

The universal M-bundle $E_M \to B\operatorname{Diff}^+(M)$ gives a map

$$\kappa_{E_M} : H^*(BSO(n);\mathbb{Q}) \to H^{*-n}(B\operatorname{Diff}^+(M);\mathbb{Q}).$$

The homomorphism κ_E is natural in the sense that $h^* \circ \kappa_E = \kappa_{h^*E}$ for any map h and so the images of κ_E can be viewed as characteristic classes of manifold bundles, which we call *generalized Morita-Miller-Mumford classes* or short MMM-classes. Morita [20], Miller [19] and Mumford [22] first studied these classes in the 2-dimensional case.

For a graded vector space V and $n \in \mathbb{N}$, we denote by $\sigma^{-n}V$ the new graded vector space with $(\sigma^{-n}V)_m = 0$ if $m \leq 0$ and $(\sigma^{-n}V)_m = V_{m+n}$ for $m > 0$. Then κ_E becomes a map $\sigma^{-n}H^*(BSO(n);\mathbb{Q}) \to H^*(B;\mathbb{Q})$ of graded vector spaces.

Let $\mathcal{R}_n$ be a set of representatives for the oriented diffeomorphism classes of oriented closed n-manifolds (connected or non-connected) and let $\mathcal{R}_n^0 \subset \mathcal{R}_n$ be the set of connected n-manifolds. Put

$$\mathcal{B}_n := \coprod_{M \in \mathcal{R}_n} B\operatorname{Diff}^+(M);\ \mathcal{B}_n^0 = \coprod_{M \in \mathcal{R}_n^0} B\operatorname{Diff}^+(M) \subset \mathcal{B}_n.$$

There are tautological smooth fibre bundles on these spaces and therefore we get maps of graded vector spaces

$$\kappa^n : \sigma^{-n}H^*(BSO(n);\mathbb{Q}) \to H^*(\mathcal{B}_n;\mathbb{Q});\ \kappa^{n,0} : \sigma^{-n}H^*(BSO(n);\mathbb{Q}) \to H^*(\mathcal{B}_n^0;\mathbb{Q}); \tag{1.3}$$

$\kappa^{n,0}$ is the composition of κ^n with the restriction map $H^*(\mathcal{B}_n) \to H^*(\mathcal{B}_n^0)$. Here is our first main result.

Theorem A.
1. *For even n, $\kappa^{n,0} : \sigma^{-n}H^*(BSO(n);\mathbb{Q}) \to H^*(\mathcal{B}_n^0;\mathbb{Q})$ is injective.*
2. *For odd n, the kernel of $\kappa^{n,0} : \sigma^{-n}H^*(BSO(n);\mathbb{Q}) \to H^*(\mathcal{B}_n^0;\mathbb{Q})$ is the linear subspace that is generated by the components $\mathcal{L}_{4d} \in H^{4d}(BSO(n);\mathbb{Q})$ of the Hirzebruch $\mathcal{L}$-class (for $4d > n$).*

Equivalently, Theorem A says (for even n) that for each $0 \neq c \in \sigma^{-n} H^*(BSO(n);\mathbb{Q})$, there is a connected n-manifold M and a smooth oriented M-bundle $f : E \to B$ such that $\kappa_E(c) \neq 0 \in H^*(B)$. Similarly for odd n.

Generalized MMM-classes of degree 0 are also interesting: these are just the characteristic numbers of the fibre. The linear independence of those is a well-known classical result by Thom [25] and therefore we only care about positive degrees.

For an arbitrary graded $\mathbb{Q}$-vector space V (concentrated in positive degrees), we let ΛV be the free graded-commutative unital $\mathbb{Q}$-algebra generated by V. If A is a graded-commutative $\mathbb{Q}$-algebra, then any graded vector space homomorphism $\phi : V \to A$ extends uniquely to an homomorphism $\Lambda\phi : \Lambda V \to A$ of graded algebras such that $\Lambda\phi \circ s = \phi$ where $s : V \to \Lambda V$ is the natural inclusion. Therefore, the map κ^n from 1.3 induces a homomorphism

$$\Lambda\kappa^n : \Lambda\sigma^{-n} H^*(BSO(n);\mathbb{Q}) \to H^*(\mathcal{B}_n;\mathbb{Q}). \tag{1.4}$$

Our second main result is about $\Lambda\kappa^n$.

Theorem B. (1) *If n is even, then the map $\Lambda\kappa^n$ from 1.4 is injective.*

(2) *If n is odd, then the kernel of $\Lambda\kappa^n$ is the ideal generated by the components $\mathcal{L}_{4d} \in H^{4d}(BSO(n);\mathbb{Q})$ of the Hirzebruch $\mathcal{L}$-class (for $4d > n$).*

We show a similar result in the complex case. A holomorphic fibre bundle of dimension m is a proper holomorphic submersion $f : E \to B$ between complex manifolds of codimension $-m$. By Ehresmann's fibration theorem, f is a smooth oriented fibre bundle (but the biholomorphic equivalence class of the fibres is not locally constant). The vertical tangent bundle $T_vE := \ker Tf$ is a complex vector bundle of rank n and for any $c \in H^*(BU(m))$, we can define

$$\kappa_E^{\mathbb{C}}(c) := f_!(c(T_vE)) \in H^{*-2m}(B).$$

Theorem C. (1) *For each $0 \neq c \in \sigma^{-2m} H^*(BU(m))$, there exists a holomorphic fibre bundle $f : E \to B$ of dimension m on a projective variety B such that $f_!(c(T_vE)) \neq 0$.*

(2) *For any $0 \neq c \in \Lambda\sigma^{-2m} H^*(BU(m))$, there exists a holomorphic fibre bundle with m-dimensional fibres on an open complex manifold such that $\Lambda\kappa_E^{\mathbb{C}}(c) \neq 0$.*

Note that it is far from obvious to say what the universal holomorphic bundle should be. Therefore we do not formulate Theorem C in the language of universal bundles.

The results of this paper can be interpreted in the language of the Madsen-Tillmann-Weiss spectra MTSO(n) [10], as we will briefly explain. By definition, MTSO(n) is the Thom spectrum of the inverse of the universal vector bundle $L_n \to BSO(n)$. If $f : E \to B$ is an oriented manifold bundle of fibre dimension n, then the Pontrjagin-Thom construction yields a spectrum map $\alpha^\flat : \Sigma^\infty B_+ \to$ MTSO(n). The spectrum cohomology of MTSO(n) is, by the Thom isomorphism,

isomorphic to $H^{*+n}(BSO(n))$. Therefore, $\alpha^\flat$ induces a map of graded vector spaces $\sigma^{-n}H^*(BSO(n)) \to H^*(B)$, which is the same as the map κ_E.

The adjoint of $\alpha^\flat$ is a map $\alpha : B \to \Omega_0^\infty \mathrm{MTSO}(n)$ and it induces an algebra map $H^*(\Omega_0^\infty \mathrm{MTSO}(n)) \to H^*(B)$. Under the classical isomorphism $H^*(\Omega_0^\infty \mathrm{MTSO}(n); \mathbb{Q}) \cong \Lambda H^{*>0}(\mathrm{MTSO}(n); \mathbb{Q})$, this map corresponds to Λ. Apart from the breakthrough works [16], [17], [10], the characteristic classes of manifold bundles related to $\mathrm{MTSO}(n)$ have been studied by several authors [11], [24]. Their methods, however, do not suffice to show Theorems A and B.

In general, the construction of manifold bundles and the computation of generalized MMM-classes are rather difficult problems. The only difficult constructions which we need in the present paper are in the 2-dimensional case, and for that we rely entirely on [19] and [20]. There are some other computations of MMM-classes which we want to mention though we do not need them.

The MMM-classes of bundles with compact connected Lie groups as structural groups are relatively easy to compute due to the "localization formula" of [3]. A special case is the case of homogeneous space bundles of the form $BH \to BG$ where $H \subset G$ are compact Lie groups. In that case, the MMM-classes can be expressed entirely in terms of Lie-theoretic data. In [1], a similar localization principle is applied to cyclic structural groups.

The MMM-classes associated with *multiplicative sequences* are rather well understood because of the close relationship with genera (i.e., ring homomorphisms from the oriented bordism ring to $\mathbb{Q}$), see e.g. [13]. The theory of elliptic genera shows that many of these MMM-classes are nontrivial. Unfortunately, this is not enough to establish Theorem A.

Another source of smooth fibre bundles with nontrivial MMM-classes is the following result. If M is an oriented manifold with signature 0, then there exists an oriented smooth fibre bundle $E \to \mathbb{S}^1$ such that E is oriented cobordant to M. This was established by Burdick and Conner (combine Corollary 6.3 of [7] with Theorem 1.2 of [6]) away from the prime 2. Another proof was given by W. Neumann [23] based on a result of Jänich [14]. Let $0 \neq x \in H^{4k}(BSO(4k))$ be a class that is not a multiple of the Hirzebruch class. Then there is a $4k$-manifold M with signature 0 and $\langle x(TM); [M] \rangle \neq 0$ and a fibre bundle $f : M \to \mathbb{S}^1$ by the above results. Then $f_!(x(T_v M)) \neq 0 \in H^1(\mathbb{S}^1)$. Therefore, in all dimensions of the form $4k-1$, the statement of Theorem 2.3 is true for classes of degree 1.

In section 2, we give a detailed overview of the proof of the main results. In the appendix, we recapitulate the definitions and the relevant properties of the Gysin-homomorphism and the related transfer. The rest of the paper contains the details of the proof outlined in section 2.

Acknowledgement. The author wants to acknowledge the hospitality and generosity of the Mathematics Department of the University of Copenhagen, which is where this project was begun.

Notations and conventions. All cohomology groups in this paper have rational coefficients. When G is a topological group which acts on the space X, we denote the Borel construction by $E(G;X) := EG \times_G X$. We furthermore abbreviate $\mathcal{B}_n := \coprod_{M \in \mathcal{R}_n} B\operatorname{Diff}^+(M)$. Our notation of standard characteristic classes differs from the customary one. We give them the actual cohomological degree they have as an index. For an example, $p_4(V)$ will denote what is commonly known as the first Pontrjagin class of the real vector bundle. We hope that this does not lead to confusion. If x is an element of a graded vector space, we denote its degree by $|x|$, implicitly assuming that x is homogeneous. Moreover, all sub vector spaces $W \subset V$ of a graded vector space are assumed to be graded, in other words $W = \oplus_n W \cap V_n$. The dual space of a vector space V is always denoted by $V^\vee$.

2. Outline of the proof

The proof of Theorems A, B and C is an eclectic combination of several computations. In this section, we give an outline. The cohomology of $BSO(n)$ is well known:

$$H^*(BSO(2m+1);\mathbb{Q}) \cong \mathbb{Q}[p_4,\ldots,p_{4m}];\ H^*(BSO(2m);\mathbb{Q}) \cong \mathbb{Q}[p_4,\ldots,p_{4m},\chi]/(\chi^2 - p_{4m}).$$

The cases $n = 0, 1$ of Theorem A are empty. The fact that the subspace generated by the components of the Hirzebruch $\mathcal{L}$-class is contained in the kernel of κ^n for odd n follows from the multiplicativity of the signature in fibre bundles of odd-dimension: If $f : E \to B$ is a smooth oriented fibre bundle with odd-dimensional fibres and B is a closed oriented manifold, then $\operatorname{sign}(E) = 0$. This was first mentioned by Atiyah [2] (without proof), proven later by Meyer [18], Lück-Ranicki [15] and the author [8]. For further reference, we state this result explicitly.

Theorem 2.1. *For odd n, the kernel of κ^n contains the subspace that is generated by the components $\mathcal{L}_{4d} \in H^{4d}(BSO(n);\mathbb{Q})$ of the Hirzebruch $\mathcal{L}$-class (for $4d > n$).*

Theorem A shows that this is the only constraint. Because the components of $\mathcal{L}$ form an additive basis of $H^*(BSO(3);\mathbb{Q})$, Theorem 2.1 forces κ^3 to be the zero map. Thus Theorem A is also empty in the 3-dimensional case. The case $n = 2$ is a classical result, which is the main ingredient for the proof of Theorem A.

Theorem 2.2. *The map $\kappa^{2,0}$ is injective.*

This was first established by Miller [19] and Morita [20]. Today, there are other proofs by Akita-Kawazumi-Uemura [1] and Madsen-Tillmann [16]. Of course, the affirmative solution of the Mumford conjecture by Madsen and Weiss [17] also implies Theorem 2.2.

We denote by $\operatorname{Pont}^*(n) \subset H^*(BSO(n);\mathbb{Q})$ the subring generated by the Pontrjagin classes. If $V \to X$ is a real vector bundle, then $\operatorname{Pont}(V) \subset H^*(X)$ is the subring generated by the Pontrjagin classes of X. The main bulk of work to prove Theorem A is:

Theorem 2.3. (1) *For even n, $\kappa^{n,0} : \sigma^{-n}\operatorname{Pont}^*(n) \to H^*(\mathcal{B}_n^0)$ is injective.*

(2) *For odd* n, *the kernel of* $\kappa^{n,0}: \sigma^{-n}\operatorname{Pont}^*(n) \to H^*(\mathcal{B}_n^0)$ *is the linear subspace that is generated by* $\mathcal{L}_{4d}$ *(for* $4d > n$*).*

Theorem 2.3 implies Theorem B: for odd n, $\operatorname{Pont}^*(n) = H^*(BSO(n))$ and for $n = 2m$, the argument is so short and easy that we give it here. The total space of the unit sphere bundle of the universal vector bundle on $BSO(2m+1)$ is homotopy equivalent to $BSO(2m)$ and the bundle projection corresponds to the inclusion map $f: BSO(2m) \to BSO(2m+1)$. This map induces an isomorphism $\operatorname{Pont}^*(2m+1) \to \operatorname{Pont}^*(2m)$. Any element $x \in H^*(BSO(2m);\mathbb{Q})$ can be written uniquely as $x = f^*x_1\chi + f^*x_2$ with $x_i \in H^*(BSO(2m+1);\mathbb{Q})$. Lemma 2.4 below and Theorem 2.3 immediately imply Theorem B.

Lemma 2.4. *Let* $f: BSO(2m) \to BSO(2m+1)$ *be the universal* $\mathbb{S}^{2m}$*-bundle and let* $x = f^*x_1\chi + f^*x_2$ *be as above. Then* $p_!(x(T_vBSO(2m))) = 2x_1$*.*

Proof. The vertical tangent bundle $T_vBSO(2m)$ is isomorphic to the universal $2m$-dimensional vector bundle. Therefore: $f_!(x(T_v(E))) = f_!(x) = f_!(f^*x_1\chi + f^*x_2) = x_1f_!(\chi) + f_!(1)x_2 = 2x_1$, since $f_!(\chi) = \chi(\mathbb{S}^{2m}) = 2$ and $f_!(1) = 0$. □

The proof of Theorem 2.3 has two parts. The first part is an induction argument, using Theorem 2.2 as induction beginning and the second part deals with the classes that are missed by the inductive argument. The idea of the induction is straghtforward. Let n be given. Let $f_i: E_i \to B_i$ be manifold bundles of fibre dimension n_i, $i = 1,2$, $n_1 + n_2 = n$. The idea is to consider the product bundle $f = f_1 \times f_2 : E_1 \times E_2 \to B_1 \times B_2$, which has fibre dimension n. The MMM-classes of the product can be expressed by the MMM-classes of the two factors. It turns out that we can detect most, but not all MMM-classes on products of lower-dimensional manifold bundles. Here is the exception.

Recall that the *Pontrjagin character* of a real vector bundle $V \to X$ is $\operatorname{ph}(V) := \operatorname{ch}(V \otimes \mathbb{C})$. Since $V \otimes \mathbb{C} \cong \overline{V \otimes \mathbb{C}}$ (it is self-conjugate), it follows that $\operatorname{ph}_{4d+2}(V) = 0$, so ph is concentrated in degrees that are divisible by 4. In fact, $\operatorname{ph}_{4d} \in \operatorname{Pont}^{4d}(n)$, $n = \operatorname{rank}(V)$. Note that if V is itself complex, then $\operatorname{ph}(V) = \operatorname{ch}(V \otimes_{\mathbb{R}} \mathbb{C}) = \operatorname{ch}(V \oplus \overline{V})$.

Proposition 2.5. (1) *Let* $n = 2m$ *be even and assume that Theorem 2.3 has been proven for all even dimensions* $2l < n$*. Then the kernel of* $\kappa^{n,0}: \sigma^{-n}\operatorname{Pont}^*(n) \to H^*(\mathcal{B}_n^0;\mathbb{Q})$ *is contained in the span of the components* ph_{4d}*,* $4d \geq n$*.*

(2) *Let* $n = 2m+1 \geq 7$ *be odd and assume that Theorem 2.3 has been proven for all dimensions less than* n*. Then the kernel of* $\kappa^{n,0}: \sigma^{-n}\operatorname{Pont}^*(n) \to H^*(\mathcal{B}_n^0;\mathbb{Q})$ *is contained in the span of the components* ph_{4d} *and* $\mathcal{L}_{4d}$*,* $4d \geq 2m+1$*.*

The proof is given in section 3. By Proposition 2.5 and Theorem 2.2, two steps remain to be done for the proof of Theorem 2.3 and hence Theorem A. We have to show that $\kappa^n(\operatorname{ph}_{4d}) \neq 0$ for all $4d \geq n \geq 4$. Furthermore, we have to show the case $n = 5$ of Theorem 2.3 from scratch.

There are two ideas involved: we do explicit computations for bundles of complex projective spaces and then we use what we call "loop space construction" to increase the dimension.

Let the group $SU(m+1)$ act on $\mathbb{CP}^m$ in the usual way. Consider the Borel-construction $q : E(SU(m+1); \mathbb{CP}^m) \to BSU(m+1)$. In section 5, we will show the following result.

Theorem 2.6. *For all $d \geq k$, the class $\kappa_{E(SU(2k+1);\mathbb{CP}^{2k})}(\mathrm{ph}_{4d}) \in H^{4d-4k}(BSU(2k+1))$ is nonzero.*

To finish the proof of Theorem A in the even-dimensional case it remains to prove that $\kappa^{4k+2}(\mathrm{ph}_{4d}) \neq 0$ if $4d \geq 4k+2 \geq 6$. To do this, we employ the loop space construction that we describe now.

Let M be an oriented closed n-manifold and $f : E \to X$ a smooth oriented M-bundle. Let LX be the free loop space of X and let $\mathrm{ev} : \mathbb{S}^1 \times LX \to X$ be the evaluation map $\mathrm{ev}(t, \gamma) := \gamma(t)$.

Consider the diagram (pr is the obvious projection):

$$\begin{tikzcd}
\mathfrak{L}E := \mathbb{S}^1 \times LX \times_X E \arrow[r, "h"] \arrow[d, "f'"] & E \arrow[d, "f"] \\
\mathbb{S}^1 \times LX \arrow[r, "\mathrm{ev}"] \arrow[d, "\mathrm{pr}"] & X \\
LX &
\end{tikzcd} \tag{2.7}$$

The composition on the left-hand side is denoted $\mathfrak{L}p := \mathrm{pr} \circ f' : \mathfrak{L}E \to LX$; this is a smooth oriented $\mathbb{S}^1 \times M$-bundle. We call it the *loop space construction* on the bundle E.

The generalized MMM-classes of $\mathfrak{L}E \to LX$ can be expressed in terms of those of $E \to X$. The result is that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Pont}^*(n+1) \arrow[r] \arrow[d, "\kappa_{\mathfrak{L}E}"] & \mathrm{Pont}^*(n) \arrow[d, "\kappa_E"] \\
H^{*-n-1}(LX) & H^{*-n}(X). \arrow[l, "\mathrm{trg}"']
\end{tikzcd} \tag{2.8}$$

The bottom map is the *transgression*, see Definition 4.3. We can of course iterate the loop space construction. When we apply it r times to the M-bundle $E \to X$, we obtain an $(\mathbb{S}^1)^r \times M$-bundle $\mathfrak{L}^r p : \mathfrak{L}^r E \to L^r(X) = \mathrm{map}((\mathbb{S}^1)^r; X)$. Also, the transgression can be iterated and gives $\mathrm{trg}^r : H^*(X) \to H^{*-r}(L^r X)$. Now let $f : E \to X$ be an M^{4k}-bundle and let $4d \geq 4k + r$. Assume that $f_!(\mathrm{ph}_{4d}(T_v E)) \in H^{4d-4k}(X)$ is nonzero. Since ph_{4d} does not lie in the kernel of the restriction $H^*(BSO(4d+r)) \to H^*(BSO(4n))$, the class $\kappa^{\mathfrak{L}^r E}(\mathrm{ph}_{4d}) \in H^{4d-4k-r}(L^r X)$ is nontrivial provided that $f_!(\mathrm{ph}_{4d}(T_v E)) \in H^{4d-4k}(X)$ does not lie in the kernel of trg^r.

For a general space X, the transgression is far from being injective, but it is injective if X is simply-connected and the rational cohomology of X is a free graded-commutative algebra, compare 4.6. If X is an addition r-connected, then trg^r is injective.

The base space $BSU(2k+1)$ of the universal $\mathbb{CP}^{2k}$-bundle in Theorem 2.6 is 3-connected and its rational cohomology is a polynomial algebra and so trg^r is injective for $r = 1, 2, 3$. Therefore, Theorem 2.6 implies that $\kappa^n(\mathrm{ph}_{4d}) \neq 0$ if $n = 4k + r$ for $0 \leq r \leq 3$. This concludes, by Proposition 2.5, the proof of Theorem A in the even-dimensional case.

For the odd-dimensional case, the only thing that is left is the induction beginning (in dimension 5). This is accomplished by the same method.

Theorem 2.9. *Let $q : E(SU(3); \mathbb{CP}^2) \to BSU(3)$ be the Borel construction and $d > 0$. Then the kernel of $\kappa_{E(SU(3);\mathbb{CP}^2)} : \mathrm{Pont}^{4d+4}(4) \to H^{4d}(BSU(3))$ is one-dimensional and spanned by $\mathcal{L}_{4d+4}$.*

Corollary 2.10. *Let $\mathfrak{L}q : \mathfrak{L}E := \mathfrak{L}E(SU(3); \mathbb{CP}^2) \to LBSU(3)$ (it is an $\mathbb{S}^1 \times \mathbb{CP}^2$-bundle). Then the kernel of $(\mathfrak{L}q)_! : \mathrm{Pont}^{*+5}(T_v\mathfrak{L}E) \to H^*(LBSU(3))$ is spanned by the components of the Hirzebruch class.*

The corollary follows immediately from Theorem 2.9, diagram 2.8 and Proposition 4.6. This gives the induction beginning and finishes the proof of Theorem A. Actually, it is quite surprising that a single 5-manifold, namely $\mathbb{S}^1 \times \mathbb{CP}^2$, sufficed to detect all MMM-classes.

Once Theorem A is shown, Theorem B is a rather formal consequence that uses the Barratt-Priddy-Quillen Theorem on the infinite symmetric group. We will not give any details here and refer to section 6 instead.

The proof of Theorem C is a simple variation of the proofs of Theorems A and B and will be discussed in section 7.

3. The induction step

In this section, we prove Proposition 2.5. First we recall that the Hirzebruch $\mathcal{L}$-class is the multiplicative sequence in the Pontrjagin classes that is associated with the formal power series

$$\sqrt{x}\coth(\sqrt{x}) = \sum_{d=0}^{\infty} \frac{2^{2k}B_{2k}}{(2k)!}x^k, \tag{3.1}$$

where B_{2k} denote the Bernoulli numbers. It is crucial for our proofs that $B_{2k} \neq 0$.

The main part of the proof is pure linear algebra. The Whitney sum map $BSO(n_1) \times BSO(n_2) \to BSO(n)$ $(n_1 + n_2 = n)$ induces a map

$$r_{n_1,n_2} : \sigma^{-n}\,\mathrm{Pont}^*(n) \to \sigma^{-n_1}\,\mathrm{Pont}^*(n_1) \otimes \sigma^{-n_2}\,\mathrm{Pont}^*(n_2).$$

Furthermore, we let $L(n) \subset \sigma^{-n} \operatorname{Pont}^*(n)$ be the subspace spanned by the components of the Hirzebruch $\mathcal{L}$-class. For $n_1 + n_2 = n$, let $\tilde{r}_{n_1,n_2}$ be the composition

$$\tilde{r}_{n_1,n_2} : \sigma^{-n} \operatorname{Pont}^*(n) \stackrel{r_{n_1,n_2}}{\to} \sigma^{-n_1} \operatorname{Pont}^*(n_1) \otimes \sigma^{-n_2} \operatorname{Pont}^*(n_2) \to \sigma^{-n_1} \operatorname{Pont}^*(n_1) \otimes (\sigma^{-n_2} \operatorname{Pont}^*(n_2))/L(n_2)$$

with the quotient map.

Lemma 3.2. (1) *Let $n = 2m$. Then the intersection $\bigcap_{m_1+m_2=m, 0<m_1<m} \ker(r_{2m_1,2m_2}) \subset \sigma^{-n} \operatorname{Pont}^*(n)$ is the vector space spanned by the elements ph_{4d}, $4d \geq n$.*

(2) *Let $n = 2m+1 \geq 7$. Then the intersection $\bigcap_{m_1+m_2=m, 0<m_1<m} \ker(\tilde{r}_{2m_1,2m_2+1}) \subset \sigma^{-n} \operatorname{Pont}^*(2m+1)$ is the vector space spanned by ph_{4d} and $\mathcal{L}_{4d}$, $4d \geq n$.*

Proof of Lemma 3.2, part 1. We identify $\operatorname{Pont}^*(2m) = \mathbb{Q}[x_1, \ldots, x_m]^{\Sigma_m}$, where $x_1, \ldots, x_m$ are indeterminates of degree 4, the Pontrjagin classes correspond to elementary symmetric functions and ph_{4d} to $x_1^d + \ldots + x_m^d$. Let us introduce some abbreviations. If $S = \{i_1, \ldots, i_s\} \subset \underline{m}$, then $V_S := \mathbb{Q}[x_{i_1}, \ldots, x_{i_s}]$. Moreover, $V_S^{<d}$ denotes the subspace of element of degree less than d (and, as usual, all degrees are total degrees).

The kernel of $r_{2k,2n-2k}$ agrees, up to a degree shift, with the kernel of the quotient map

$$V_{\{1,\ldots,m\}} \to \frac{V_{\{1,\ldots k\}} \otimes V_{\{k+1,\ldots,m\}}}{V^{<2k}_{\{1,\ldots k\}} \otimes V_{\{k+1,\ldots,m\}} \oplus V_{\{1,\ldots k\}} \otimes V^{<2m-2k}_{\{k+1,\ldots,m\}}}$$

which is the same as

$$\bigcap_{k=1}^{m-1} V^{<2k}_{\{1,\ldots,k\}} \otimes V_{\{k+1,\ldots,m\}} \oplus V_{\{1,\ldots,k\}} \otimes V^{<2m-2k}_{\{k+1,\ldots,m\}}. \tag{3.3}$$

Let $4d \geq 2m$. We have to show the following: if a homogeneous symmetric polynomial $p(x_1 \ldots, x_m)$ of degree $4d$ lies in the intersection 3.3, then p must be a power sum, i.e. a multiple of ph_{4d}.

Let $\mathcal{P}$ is the set of all partitions of the set $\underline{m}$ into two parts, $\underline{m} = S_1 \coprod S_2$. A *symmetric* polynomial p lies in the intersection 3.3 if and only if it lies in

$$U = \bigcap_{P \in \mathcal{P}} V_{S_1}^{<2|S_1|} \otimes V_{S_2} \oplus V_{S_1} \otimes V_{S_2}^{<2|S_2|}. \tag{3.4}$$

Clearly, each of the spaces in 3.4 whose intersection is U is spanned by monomials. Therefore U is spanned by monomials, too. Therefore, the space U has the following property: if $p \in U$ is written as a linear combination of monomials $p = \sum_i a_i p_i$ with pairwise distinct monomials p_i and $0 \neq a_i \in \mathbb{Q}$, then $p_i \in U$.

We call the monomials of the form x_i^j *pure* and all the other ones *impure*. We will show that any monomial in U is pure. This finishes the proof because then any

symmetric $p \in U$ must be a linear combination of pure monomials and the symmetry forces p to be a power sum.

Let now p be an impure monomial of degree $4d \geq 2m$. We want to show that p does not lie in U. Without loss of generality (symmetry!), we can assume that $p = x_1^{d_1} \ldots x_k^{d_k}$ and $0 < d_k \leq d_j$ for all $j = 1, \ldots, k$. Note that $k \geq 2$ since p is impure. There are three cases to distinguish.

- If $k = m$, then $4(d - d_k) = 4(d_1 + \ldots + d_{k-1}) \geq 4(k-1)d_k \geq 4(k-1) \geq 2(m-1)$. Then f is not contained in the space
$$V^{<2(m-1)}_{\{1,\ldots,m-1\}} \otimes V_{\{m\}} \oplus V_{\{1,\ldots,m-1\}} \otimes V^{<2}_{\{m\}}$$
and hence not in U.
- If $k < m$ and $2d_k \geq m - k$, then $4(d - d_k) \geq 4(k-1)d_k \geq 2(k-1)(m-k) \geq 2(k-1)$. Then f is not contained in
$$V^{<2(k-1)}_{\{1,\ldots k-1\}} \otimes V_{\{k,\ldots,m\}} \oplus V_{\{1,\ldots,k-1\}} \otimes V^{<2(m-k)}_{\{k,\ldots m\}}$$
and hence not in U.
- If $k < m$ and $2d_k < m - k$. Put $e = 2d_k$. Then $2d_k \geq e$ and $2(d - d_k) = 2d - e \geq m - e$ and thus f does not lie in
$$V^{<2(m-e)}_{\{1,\ldots k-1,k+1,\ldots m-e+1\}} \otimes V_{\{k,m-e+2,\ldots,m\}} \oplus$$
$$V_{\{1,\ldots k-1,k+1,\ldots m-e+1\}} \otimes V^{<2e}_{\{k,m-e+2,\ldots,m\}}$$
and hence it is not in U either.

$\square$

To show the second part of Lemma 3.2, we need another lemma.

Lemma 3.5. *Let $f = \sum_{k=0}^{\infty} f_k x^k \in 1 + x\mathbb{Q}[[x]]$ be a power series such that $f_k \neq 0$ for all k. Let $F = \sum_{i \geq 0} F_i$ be the corresponding multiplicative sequence. Let $m \geq 3$ and let $h(x_1, \ldots, x_m)$ be a symmetric homogeneous polynomial of degree d. Assume that*

$$h(x_1, \ldots, x_m) = \sum_{i=0}^{d} a_i x_m^i F_{d-i}(x_1, \ldots, x_{m-1}).$$

Then $h(x_1, \ldots, x_m) = a_0 F_d(x_1, \ldots, x_m)$.

Proof. The assumption that $f_k \neq 0$ implies that F_i is nonzero (for all i and an arbitrary positive number of variables). By definition of multiplicative sequences, we can write

$$h(x_1, \ldots, x_m) = \sum_{k=0}^{d} \sum_{i+j=k} a_i f_j x_m^i x_{m-1}^j F_{d-k}(x_1, \ldots, x_{m-2})$$

and by symmetry we also have

$$h(x_1,\dots,x_m)=\sum_{k=0}^{d}\sum_{i+j=k}a_if_jx_{m-1}^ix_m^jF_{d-k}(x_1,\dots,x_{m-2}).$$

Therefore $a_if_j=a_jf_i$ for all $0\le i+j\le d$ (here the assumption that $m\ge 2$ is essential). Thus $a_j=a_jf_0=f_ja_0$ and hence

$$h(x_1,\dots,x_m)=a_0\sum_{i=0}^{d}f_ix_m^iF_{d-i}(x_1,\dots,x_{m-1})=a_0F_d(x_1,\dots,x_m).$$

□

Proof of Lemma 3.2, Part 2. The space $\ker(\tilde r_{2m_1,2m_2+1})\subset\sigma^{-n}\operatorname{Pont}^*(2m+1)$ is the sum of the space $\ker(r_{2m_1,2m_2+1})$ (which was computed in the proof of the first part) and the space $\mathbb{Q}[x_1,\dots,x_{m_1}]\otimes L(2m_2+1)$. There is an inclusion relation $\mathbb{Q}[x_1]\otimes L(2m-1)\subset\mathbb{Q}[x_1,\dots,x_{m_1}]\otimes L(2m_2+1)$ for all $m_1\ge 1$. Thus the intersection agrees with the smallest space, i.e. $\mathbb{Q}[x_1]\otimes L(2m-1)$. A homogeneous symmetric polynomial $f\in\mathbb{Q}[x_1]\otimes L(2m-1)$ of degree $4d$ can be written as $f(x_1,\dots,x_m)=\sum_{i=0}^{d}a_ix_1^i\mathcal{L}_{4i}(x_2,\dots,x_m)$ with $a_i\in\mathbb{Q}$. By Lemma 3.5, f is a multiple of the Hirzebruch class (here we use that the coefficients of the power series 3.1 are nonzero). □

Proof of Proposition 2.5. Let $f_i:E_i\to B_i$, $i=1,2$, be two oriented fibre bundles of fibre dimension $n_i>0$ with $n_1+n_2=n$. Consider $f=f_1\times f_2:E=E_1\times E_2\to B=B_1\times B_2$, which is an oriented fibre bundle of fibre dimension n. The umkehr homomorphism is compatible with products in the sense that

$$(f_1\times f_2)_!(x_1\times x_2)=(f_1)_!(x_1)\times(f_2)_!(x_2)\tag{3.6}$$

for all $x_i\in H^*(E_i)$ (the signs are all $+1$ since x_i has even degree). Therefore the diagram

$$\begin{array}{ccc}\sigma^{-n}\operatorname{Pont}^*(n) & \xrightarrow{\kappa^{n,0}} & H^*(\mathcal{B}_n^0)\\ \downarrow{\scriptstyle r_{n_1,n_2}} & & \downarrow\\ \sigma^{-n_1}\operatorname{Pont}^*(n_1)\otimes\sigma^{-n_2}\operatorname{Pont}(n_2) & \xrightarrow{\kappa^{n_1,0}\otimes\kappa^{n_2,0}} & H^*(\mathcal{B}_{n_1}^0)\otimes H^*(\mathcal{B}_{n_2}^0)\end{array}\tag{3.7}$$

is commutative; the left-hand side vertical map is induced by the Whitney sum and the right hand side vertical map is induced by taking product bundles. A straightforward application of Lemma 3.2 completes the proof. □

4. The loop space construction

The loop space construction. Let M be an oriented closed n-manifold and $f : E \to X$ a smooth oriented M-bundle. Let LX be the free loop space of X and let $\mathrm{ev} : \mathbb{S}^1 \times LX \to X$ be the evaluation map $\mathrm{ev}(t,\gamma) := \gamma(t)$. Moreover, $\eta : LX \to X$ is defined by $\eta(\gamma) = \mathrm{ev}(1,\gamma)$. Recall that the loop space construction $\mathfrak{L}p = \mathrm{pr} \circ f'$ is defined by the diagram 2.7:

(4.1)
$$\begin{tikzcd} \mathfrak{L}E := \mathbb{S}^1 \times LX \times_X E \arrow[r, "h"] \arrow[d, "f'"] & E \arrow[d, "f"] \\ \mathbb{S}^1 \times LX \arrow[r, "\mathrm{ev}"] \arrow[d, "\mathrm{pr}"] & X \\ LX. & \end{tikzcd}$$

Relation to loop groups. There is an alternative view on the loop space construction which might be illuminating though we do not need it in the sequel.

Let G be a topological group and let M be an oriented closed n-manifold with a G-action. Let $f : E = E(G;M) := EG \times_G M \to X = BG$ be the Borel construction, an oriented M-bundle. The loop group LG (multiplication is defined pointwise) acts on $\mathbb{S}^1 \times M$ by the formula

$$\gamma \cdot (t,m) := (t, \gamma(t)m),$$

where $\gamma \in LG$, $t \in \mathbb{S}^1$, $m \in M$. Thus we get an induced $\mathbb{S}^1 \times M$-bundle $q : E(LG; \mathbb{S}^1 \times M) \to BLG$.

Given an LG-principal bundle $Q \to X$, then $(Q \times \mathbb{S}^1 \times G)/\sim \to \mathbb{S}^1 \times X$, where $(q,t,g) \sim (q\gamma, t, \gamma(t)g)$ for $\gamma \in LG$, is a G-principal bundle. In the universal case $X = BLG$, the classifying map of this bundle is a map $\phi : BLG \to LBG$, which is a homotopy equivalence if G is connected. It is not hard to see that there is a pullback-diagram

$$\begin{tikzcd} E(LG;\mathbb{S}^1 \times M) \arrow[r] \arrow[d, "q"] & \mathfrak{L}E(G;M) \arrow[d, "\mathfrak{L}p"] \\ BLG \arrow[r, "\phi"] & LBG. \end{tikzcd}$$

Characteristic classes of the loop space construction. Let us compute the generalized MMM-classes of the bundle $\mathfrak{L}f : \mathfrak{L}E \to LX$ of 2.7 in terms of those of the original bundle $f : E \to X$. Let $x \in \mathrm{Pont}^*(n+1)$. We denote the vertical tangent bundles by $T^{\mathfrak{L}f}$ and T^f in self-explaining notation. The vertical tangent bundle of $\mathfrak{L}f = \mathrm{pr} \circ f'$ is seen to be isomorphic to $(f')^* T^{\mathrm{pr}} \oplus T^{f'} \cong \mathbb{R} \oplus h^* T^f$. Therefore

(4.2)
$$(\mathfrak{L}f)_!(x(T^{\mathfrak{L}f}) = (\mathfrak{L}f)_!(x(h^*T^f)) = \mathrm{pr}_! f'_!(x(h^*T^f)) = \mathrm{pr}_! f'_! h^*(x(T^f)) = \mathrm{pr}_! \mathrm{ev}^* f_! h^*(x(T^f)),$$

using the naturality of the umkehr map. The first equation is true because $x \in \mathrm{Pont}^*(n+1)$. We can rephrase this formula using the notion of the transgression homomorphism.

Definition 4.3. Let X be a space, LX its free loop space, $\mathrm{ev} : \mathbb{S}^1 \times LX \to X$ the evaluation map and $\mathrm{pr} : \mathbb{S}^1 \times LX \to LX$ be the projection onto the first factor. The *transgression* is the homomorphism

$$\mathrm{trg} := \mathrm{pr}_! \circ \mathrm{ev}^* : H^*(X) \to H^{*-1}(LX).$$

Formula 4.2 becomes:

Proposition 4.4. *The diagram*

$$\begin{CD} \mathrm{Pont}^k(n+1) @>>> \mathrm{Pont}^k(n) \\ @V{\kappa_{\mathfrak{L}E}}VV @VV{\kappa_E}V \\ H^{k-n-1}(LX) @<{\mathrm{trg}}<< H^{k-n}(X). \end{CD} \tag{4.5}$$

is commutative.

The usefulness of the above construction stems from the fact that the transgression is injective in some cases, which we will explain now.

Proposition 4.6. *Let X be a simply-connected space such that $H^*(X;\mathbb{Q}) \cong \Lambda V$ is a free graded-commutative algebra on a finite-dimensional graded vector space V. Then $H^*(LX;\mathbb{Q})$ is a free-graded commutative algebra on a finite-dimensional vector space as well and the transgression homomorphism $\tilde{H}^*(X;\mathbb{Q}) \to H^{*-1}(LX;\mathbb{Q})$ is injective.*

Proof. Of course, the transgression is not a ring homomorphism. Instead, the following product formula holds ($\eta : LX \to X$ is the evaluation at the basepoint):

$$\mathrm{trg}(x_1 x_2) = (-1)^{|x_1|}\eta^* x_1 \,\mathrm{trg}(x_2) + \mathrm{trg}(x_1)\eta^* x_2. \tag{4.7}$$

This is shown as follows. Let $u \in H^1(\mathbb{S}^1)$ be the standard generator. Write $\mathrm{ev}^* x_i = 1 \times a_i + u \times b_i \in H^*(LX \times \mathbb{S}^1$ for some $a_i, b_i \in H^*(LX)$. Then $\mathrm{trg}(x_i) = \mathrm{pr}_! \,\mathrm{ev}^* x_i = b_i$ and $\eta^* x_i = a_i$. Formula 4.7 follows in a straightforward manner from Proposition A.3 (1).

Let $K(V^\vee) = \prod_k K(V_k^\vee; k)$ be the graded Eilenberg-Mac Lane space. There is a tautological map $s : X \to K(V^\vee)$ which induces an isomorphism in rational cohomology because the cohomology algebra of X is free graded-commutative. Therefore s is a rational homotopy equivalence. Because $\pi_1(X) = 0$, LX is simple and there is a rational homotopy equivalence $(LX)_\mathbb{Q} \simeq L(X_\mathbb{Q}) \simeq L(K(V^\vee))$. The diagram

$$\begin{CD} H^*(X;\mathbb{Q}) @>{\mathrm{trg}}>> H^{*-1}(LX;\mathbb{Q}) \\ @A{s^*}AA @AA{Ls^*}A \\ H^*(K(V^\vee);\mathbb{Q}) @>{\mathrm{trg}}>> H^{*-1}(LK(V^\vee);\mathbb{Q}) \end{CD}$$

is commutative and the vertical arrows are isomorphisms. Thus we can assume that $X = K(V^\vee)$.

The map $\eta^* \oplus \mathrm{trg} : V \oplus \sigma^{-1}V \to H^*(LX)$ induces an algebra map $\tau : \Lambda(\eta^* V \oplus \mathrm{trg}(V)) \to H^*(LX)$, which is an isomorphism by the following argument. Since a product of Eilenberg-Mac-Lane spaces is an abelian topological group, the fibration $\Omega X \stackrel{\mathrm{inc}}{\to} LX \stackrel{\eta}{\to} X$ is a product and thus it has a retraction $r : LX \to \Omega X$. The maps η^* and r^* induce an isomorphism $H^*(LX) \cong H^*(\Omega X) \otimes H^*(X)$. Moreover, it is well-known that the composition $H^*(X) \stackrel{\mathrm{trg}}{\to} H^{*-1}(LX) \stackrel{\mathrm{inc}^*}{\to} H^{*-1}(\Omega X)$ maps V to a generating subspace of the target. It follows that τ is an epimorphism which has to be an isomorphism by a dimension count.

A straightforward application of 4.7 completes the proof: let $x_1, \ldots, x_n$ be a homogeneous basis of V, $y_i := \eta^* x_i$. From 4.7, one derives the identity

$$\mathrm{trg}(x_1^{m_1} \ldots x_n^{m_n}) = \sum_{i=0}^{n} m_i y_1 \ldots y_{i-1}^{m_{i-1}} y_i^{m_i - 1} \mathrm{trg}(x_i) y_{i+1}^{m_{i+1}} \ldots y_n^{m_n}$$

which implies that trg is injective because the terms on the right hand side are all linearly independent. □

Lemma 4.8. *Let G be a simply connected compact Lie group. Then the spaces BG, LBG, L^2BG satisfy the assumptions of Proposition 4.6.*

Proof. The case of BG is a well-known result generally attributed to Borel. Since BG is 3-connected, the spaces LBG and L^2BG are simply connected and therefore the first half of the statement of Proposition 4.6 can be applied. □

5. Computations for $\mathbb{CP}^m$-bundles

Let $V \to X$ be an $(m+1)$-dimensional hermitian complex vector bundle. Let $q : \mathbb{P}(V) \to X$ be the projective bundle of V (its fibre is $\mathbb{CP}^m$ and its structural group is $\mathbb{P}U(m+1)$. The finite isogeny $SU(m+1) \to \mathbb{P}U(m+1)$ induces a rational homotopy equivalence $BSU(m+1) \to B\mathbb{P}U(m+1)$. Therefore we conclude

Lemma 5.1. *Any characteristic class of $\mathbb{CP}^m$-bundles with structural group $\mathbb{P}U(m+1)$ is a polynomial in the Chern classes (recall our indexing convention for characteristic classes) $c_4, c_6, \ldots, c_{2m+2}$ (i.e.: the first Chern class c_2 does not occur).*

We will use this Lemma in section 7. From now on, we restrict our attention to hermitian vector bundles with trivialized determinant and Lemma 5.1 tells us that we do not loose anything. There is a tautological complex line bundle $L_V \to \mathbb{P}V$ and the first Chern class of $L_V^\vee$ is denoted by $z_V \in H^2(\mathbb{P}(V))$. There is a natural isomorphism

$$T_v\mathbb{P}(V) \oplus \mathbb{C} \cong q^*V \otimes L_V^\vee. \tag{5.2}$$

Because $\mathbb{CP}^m = SU(m+1)/S(U(1)\times U(m))$, $S(U(1)\times U(m)) = SU(n+1) \cap U(1) \times U(n)$, we can identify the total space of the universal bundle $E(SU(m+1), \mathbb{CP}^m)$ with

$B(S(U(1)\times U(m)))$. Under this identification, the classifying map of the vertical tangent bundle corresponds to the map induced by the group homomorphism

$$S(U(1)\times U(m)) \to U(m) \subset SO(2m);\ \begin{pmatrix} z & 0 \\ 0 & A \end{pmatrix} \mapsto z^{-1}A. \tag{5.3}$$

The Pontrjagin character for $\mathbb{CP}^m$-bundles. In principle, the MMM-classes of the universal $\mathbb{CP}^m$-bundle $E(SU(m+1),\mathbb{CP}^m)$ were computed in Hirzebruch's lecture notes [13]. However, the formula that appears there is not appropriate to show Theorems 2.9 and 2.6. Therefore we follow another path. First we turn to the proof of Theorem 2.6, which follows immediately from Proposition 5.4 below.

Our method is to use the Leray-Hirsch Theorem for the computation of generalized MMM-classes. Let $V\to X$ be a complex vector bundle of rank $m+1$, $q:\mathbb{P}(V)\to X$, $L_V\to\mathbb{P}(V)$ and $z_V\in H^2(\mathbb{P}(V))$ as above.

As an $H^*(X)$-algebra, $H^*(\mathbb{P}V)$ is isomorphic to $H^*(X)[z_V]/(\sum_i c_{2i}(V)z_V^{m+1-i})$. The set $\{1,z_V,\dots,z_V^m\}$ is a $H^*(X)$-basis of $H^*(\mathbb{P}(V))$. Moreover, $q_!$ is the $H^*(X)$-linear map determined by $q_!(z_V^i)=0$ for $0\le i\le n-1$ and $q_!(z_V^m)=1$. The higher powers of z_V can be expressed explicitly in terms of this basis. This gives an algorithm to compute $q_!$, which is not very manageable in general. But it is manageable if all but one of the Chern classes of V are zero.

Proposition 5.4. *Let $X=BSU(2)$ and $V\to X$ the universal 2-dimensional vector bundle. Then $H^*(BSU(2))=\mathbb{Q}[u]$ where $u\in H^4(BSU(2))$ is the second Chern class of V. Consider the projective bundle $\mathbb{P}(V\oplus\mathbb{C}^{m-1})\to X$, which is a $\mathbb{CP}^m$-bundle. Then*

$$q_!(\mathrm{ch}(T_v\mathbb{P}(V\oplus\mathbb{C}^{m-1})))=\sum_{p=0}^{\infty}a_pu^p,$$

where $a_p=(-1)^p\left(\frac{m-1}{(m+2p)!}+\sum_{k+l=p}\frac{2}{(m+2k)!!(2l)!}\right)\neq 0.$

Proof. By the isomorphism 5.2, we obtain

$$q_!(\mathrm{ch}(T_v\mathbb{P}(V\oplus\mathbb{C}^{m-1})))=(\mathrm{ch}(V)+m-1)q_!(\mathrm{ch}(L^\vee_{V\oplus\mathbb{C}^{m-1}})). \tag{5.5}$$

When we write the total Chern class of V formally as $c(V)=(1+x_1)(1+x_2)$, then $x:=x_1=-x_2$ and $u=-x^2$. Thus $\mathrm{ch}(V)=\exp(x_1)+\exp(x_2)=2\cosh(x)=2\cos(\sqrt{u})$.

Let us compute $q_!(\mathrm{ch}(L^\vee_{V\oplus\mathbb{C}^{m-1}}))=\sum_{l=0}^{\infty}\frac{1}{l!}q_!(z^l_{V\oplus\mathbb{C}^{m-1}})$. With $z:=z_{V\oplus\mathbb{C}^{m-1}}$, we get the algebra isomorphism

$$H^*(\mathbb{P}(V\oplus\mathbb{C}^{m-1}))\cong\mathbb{Q}[u,z]/(z^{m+1}+uz^{m-1})$$

Therefore, for $l\ge 0$,

$$z^{m+2l+1} = (-1)^{l+1} z^{m-1} q^* u^{l+1};\; z^{m+2l} = (-1)^l z^m q^* u^l.$$

Therefore $q_!(z^{m+2l+1}) = 0$ and $q_!(z^{m+2l}) = (-1)^l u^l$ and

$$q_!(\mathrm{ch}(L^\vee_{V \oplus \mathbb{C}^{m-1}})) = \sum_{l=0}^{\infty} \frac{(-1)^l}{(m+2l)!} u^l. \tag{5.6}$$

Combine 5.6 and 5.5 to finish the proof. □

The case of $\mathbb{CP}^2$-bundles. Here we show Theorem 2.9. We consider the $\mathbb{CP}^2$-bundle $q : BS(U(1) \times U(2)) \to BSU(3)$.

Proposition 5.7. *Let $\mathcal{L}$ be the total Hirzebruch $\mathcal{L}$-class. Then $q_!(\mathcal{L}(T^q)) = 1 \in H^*(BSU(3))$. In particular, $\mathcal{L}_{4k}$ lies in the kernel of $\kappa_q : \mathrm{Pont}^{4k}(4) \to H^{4k-4}(BSU(3))$ for all $k \geq 2$.*

Proof. We offer three methods since they are all interesting. The first and most elementary method is a direct computation that can be found in [13], p.51 ff.

The second method is to use the loop space construction and then the vanishing theorem 2.1 for the resulting $\mathbb{S}^1 \times \mathbb{CP}^2$-bundle.

Another method comes also from index theory. By the family index theorem, the class $q_!(\mathcal{L}(T^q))$ agrees with the Chern character of the index bundle of the signature operator. Since the group $SU(3)$ acts by isometries on $\mathbb{CP}^2$ with respect to the Fubini-Study metric and since $SU(3)$ is connected, this index bundle is trivial. □

For the proof of Theorem 2.9, we will use complex coefficients because we are going to employ Chern-Weil theory. Let G be a compact connected Lie group with maximal torus T and Weyl group W. Let $f : BT \to BG$ be the universal G/T-bundle. Let $\mathfrak{t}$ be the Lie algebra of T and $\mathfrak{g}$ be the Lie algebra of G. Recall the Chern-Weil isomorphism $CW : \mathrm{Sym}^*(\mathfrak{g}_\mathbb{C}^\vee)^G \cong H^*(BG; \mathbb{C})$, which is natural in G. Moreover, $\mathrm{Sym}^*(\mathfrak{g}_\mathbb{C}^\vee)^G \cong \mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee)^W$ by restriction. In other words, there is a commutative diagram

$$\begin{tikzcd}
H^*(BG;\mathbb{C}) \arrow[rr, "f^*"] & & H^*(BT;\mathbb{C}) . \\
\mathrm{Sym}^*(\mathfrak{g}_\mathbb{C}^\vee)^G \arrow[u, "CW"', "\cong"] \arrow[r, "\cong", "\mathrm{res}"'] & \mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee)^W \arrow[r, "\subset"] & \mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee) \arrow[u, "CW"', "\cong"]
\end{tikzcd}$$

Now we express the transfer $\mathrm{trf}_f^* : H^*(BT;\mathbb{C}) \to H^*(BG;\mathbb{C})$ as a map $\mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee) \to \mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee)^W$.

Lemma 5.8. *As a map $\mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee) \to \mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee)^W$, the transfer trf_f^* agrees with the averaging operator $F \mapsto \sum_{w \in W} w^* F$.*

Proof. The left G-action on G/T commutes with the right-action of W. Therefore there is a fibre-preserving right-action of W on the bundle $E(G;G/T) \to BG$. The total space $E(G;G/T)$ is homotopy equivalent to BT and the homotopy equivalence is W-equivariant. In particular, $f \circ w = f$ for all $w \in W$. Therefore $\mathrm{trf}_f^* = \mathrm{trf}_{f\circ w}^* = \mathrm{trf}_f^* \circ w^*$. In other words, the transfer is W-equivariant when considered as a map $\mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee) \to \mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee)^W$. The composition

$$\mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee)^W \xrightarrow{f^*} \mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee) \xrightarrow{trf_f^*} \mathrm{Sym}^*(\mathfrak{t}_\mathbb{C}^\vee)^W$$

is the map $\mathrm{trf}_f^* f^*$, which is $\chi(G/T)\,\mathrm{id} = |W|\,\mathrm{id}$. The lemma follows from these two facts by elementary representation theory of finite groups. $\square$

Proof of Theorem 2.9. Consider the diagram

$$\begin{tikzcd}
 & BT \arrow[d, "g"] \\
BSO(4) & BS(U(1)\times U(2)) \arrow[l, "h"'] \arrow[d, "q"] \\
 & BSU(3),
\end{tikzcd}$$

where h is the classifying map of the vertical tangent bundle and T is the standard maximal torus of $SU(3)$ (the group of diagonal matrices of determinant 1). Abbreviate $f := q \circ g$. The strategy of the proof is to show first that the kernel of

$$H^{4d+4}(BSO(4)) \xrightarrow{h^*} H^{4d+4}(BS(U(1)\times U(2))) \xrightarrow{q_!} H^{4d}(BSU(3)) \tag{5.9}$$

is 2-dimensional if $d \geq 1$. The second step will be that the intersection of the kernel of 5.9 with $\mathrm{Pont}^{4d+4}(4)$ has dimension 1, generated by $\mathcal{L}_{4d+4}$, which shows the theorem.

Clearly $\dim H^{4d+4}(BSO(4)) = d+2$; we will show that the image of the composition in 5.9 has dimension d. Let $x \in H^{4d+4}(BSO(4))$. Write $x = C_1(p_4,p_8) + \chi C_2(p_4,p_8)$. Write $\mathcal{L}_{4d} = a_d p_4^d + p_8 A(p_4,p_8)$ for $a_d = 2\frac{2^{2d}B_{2d}}{(2d)!} \neq 0 \in \mathbb{Q}$ and a certain polynomial A. It follows that one can write $C_1(p_4,p_8) = a\mathcal{L}_{4d} + p_8 C_3(p_4,p_8) = a\mathcal{L}_{4d} + \chi^2 C_3(p_4,p_8)$. In other words

$$x = a\mathcal{L}_{4d+4} + \chi(\chi C_3(p_4,p_8) + C_2(p_4,p_8)) =: a\mathcal{L}_{4d+4} + \chi F(\chi,p_4)$$

for a certain polynomial. This expression is uniquely determined. Next we express $f_!(h^*(x))$ as

$$f_!(h^*(x)) = f_!(ah^*\mathcal{L}_{4d}) + f_!(h^*\chi h^* F(\chi,p_4)) = 0 + \mathrm{trf}_f^*(F(\chi,p_4))$$

by Proposition 5.7 and A.4. Therefore, the image of 5.9 agrees with the image of the composition

$$H^{4d}(BSO(4)) \stackrel{h^*}{\to} H^{4d}(BS(U(1)\times U(2))) \stackrel{\mathrm{trf}_g^*}{\to} H^{4d}(BSU(3)) \tag{5.10}$$

which is the same as image of

$$H^{4d}(BSO(4)) \stackrel{(hg)^*}{\to} H^{4d}(BS(U(1)\times U(2))) \stackrel{\mathrm{trf}_f^*}{\to} H^{4d}(BSU(3)) \tag{5.11}$$

because $\mathrm{trf}_{qg} = \mathrm{trf}_g \circ \mathrm{trf}_q$ and because $\mathrm{trf}_g^* g^*$ is the multiplication with the Euler number of the fibre of g, which is 2 since g is an $\mathbb{S}^2$-bundle.

We write the complexified Lie algebra of T as $\mathfrak{t} = \{(x_1, x_2, x_3 \in \mathbb{C}^3 | x_1 + x_2 + x_3 = 0\}$. The Weyl group is Σ_3, acting by the permutation representation. We write x_1, x_2, x_3 for the coordinate functions on $\mathfrak{t}$.

Under the map $h \circ g : BT \to BSO(4)$, the elements χ and p_1 are mapped by

$$\chi \mapsto (x_2 - x_1)(x_3 - x_1);\ p_4 \mapsto (x_2 - x_1)^2 + (x_3 - x_1)^2; \tag{5.12}$$

the reason is the isomorphism 5.2 or the equivalent expression 5.3. These elements lie in the 3-dimensional space $V := \mathrm{Sym}^2(\mathfrak{t}_{\mathbb{C}}^{\vee})$ on which we now introduce the basis

$$z_1 = (x_2 - x_1)(x_3 - x_1),\ z_2 = (x_1 - x_2)(x_3 - x_2),\ z_3 = (x_2 - x_3)(x_1 - x_3);$$

the Weyl group acts by permutations on that basis. Rewriting 5.12 yields

$$\chi \mapsto z_1; p_4 \mapsto 2z_1 + z_2 + z_3 =: z_1 + s_1; \tag{5.13}$$

where s_i denotes the W-invariant element $s_i := z_1^i + z_2^i + z_3^i$.

In view of 5.8, we have to show that the image of the $(d+1)$-dimensional subspace $X := \mathrm{span}\{z_1^k(z_1 + s_1)^{d-k}\}_{k=0,\dots,d}$ of $\mathrm{Sym}^k V$ under the averaging operator $\Phi = \sum_{\sigma \in \Sigma_3} \sigma$ has dimension d. To this end, abbreviate $v_{k,d} = z_1^k(z_1 + s_1)^{d-k}$ and note that

$$\Phi(v_{k,d}) = 2\sum_j \binom{d-k}{j} s_1^j s_{d-j}.$$

Let C be the $(d+1)\times(d+1)$-matrix with entries $c_{j,k} = \binom{d-k}{j}$ $(0 \leq j,k \leq d)$; C is nonsingular because its entries below the antidiagonal are zero and the entries on the antidiagonal are 1, therefore $\det(C) = \pm 1$. Therefore the equation

$$\Phi(\sum_k a_k v_{k,d}) = s_1^j s_{d-j}$$

has a solution (a_k) in $\mathbb{C}^{d+1}$. Therefore, the image of X under Φ contains the elements

$$s_1^d, s_1^{d-1}s_1, s_1^{d-2}s_2, \dots s_1 s_{d-1}, s_d.$$

We claim that these polynomials span an d-dimensional vector space and show this claim by induction on d. The case $d = 2$ is trivial.

Because the multiplication by s_1 is injective, it suffices to show that s_d is not a linear combination of $s_1^d, s_1^{d-1}s_1, s_1^{d-2}s_2, \dots s_1 s_{d-1}$. Assume, to the contrary that

$$s_d(z_1, z_2, z_3) = \sum_{j=0}^{d-1} c_j s_j(z_1, z_2, z_3) s_1^{d-j}(z_1, z_2, z_3), \; c_j \in \mathbb{C}.$$

Restricting to the subspace defined by $z_1 + z_2 + z_3 = 0$, we get the equation

$$z_1^d + z_2^d + (-z_3 - z_2)^d = \sum_{j=0}^{d-1} c_j s_j(z_1, z_2, z_3) s(z_1 + z_2 + z_3)^{d-j} = 0$$

which is obviously wrong for all $d \geq 2$. This finishes the proof that 5.9 has a 2-dimensional kernel.

One element in this kernel is $\mathcal{L}_{4d+4}$. Another element in $(p_4 - \chi)^{d+1}$. To see this, look at 5.13: $g^* h^*(p_4 - \chi) = s_1 \in V^{\Sigma_3} = \operatorname{Im} f^*$. Since g^* is injective, it follows that $h^*(p_4 - \chi) = q^* y$ for a certain y. It follows that

$$q_!(h^*(p_4 - \chi)^{d+1}) = q_0!(q^* y^{d+1} 1) = y^{d+1} q_!(1) = 0.$$

This means that any element in the kernel of 5.9 can be written as $a_1 \mathcal{L}_{4d+4} + a_2(p_4 - \chi)^{d+1}$. This belongs to $\mathrm{Pont}^{4d+4}(4)$ if and only if $a_2 = 0$. $\square$

6. From linear to algebraic independence

In this section, we show Theorem B, based on Theorem A whose proof we just completed. It is this step where we have to sacrifice the connectedness of the manifolds. The main step is:

Proposition 6.1. *Let $W \subset \sigma^{-n} H^*(BSO(n); \mathbb{Q})$ be a linear subspace such that $\kappa^n : W \to H^*(\coprod_{M \in \mathcal{R}} B\,\mathrm{Diff}^+(M)_+; \mathbb{Q})$ is injective. Then the extension $\Lambda\kappa^{n,0} : \Lambda W \to H^*(\coprod_{M \in \mathcal{R}} B\,\mathrm{Diff}^+(M)_+; \mathbb{Q})$ is injective.*

Assuming Proposition 6.1 for the moment, we can show Theorem B.

Proof of Theorem B: If n is even, then Theorem B is an immediate consequence of Theorem A and Proposition 6.1.

If n is odd, we need a little argument. If $W \subset V$ are graded vector spaces, then $\Lambda(V/W) \cong \Lambda(V)/(W)$, where (W) is the 2-sided ideal generated by W. Let $V := \sigma^{-n} H^*(BSO(n))$ and let W be the span of the Hirzebruch $\mathcal{L}$-classes. Choose a complement $U \subset V$ of W. By Theorem A, $\kappa^{n,0} : U \to H^*(\coprod_{M \in \mathcal{R}_n} B\,\mathrm{Diff}^+(M))$ is injective; whence $\Lambda(U) \to H^*(\coprod_{M \in \mathcal{R}_n} B\,\mathrm{Diff}^+(M))$ is injective by Proposition 6.1. But $\Lambda(U) \cong \Lambda(V)/(W)$ and therefore the statement follows. $\square$

Proof of Proposition 6.1. Without loss of generality, we can assume that W is finite-dimensional.

There exist connected n-manifolds $M_1, \ldots M_r$ such that $\kappa^n : W \to H^*(\mathcal{B}_n^0) \to H^*(\coprod_{i=1}^r B\,\mathrm{Diff}^+(M_i))$ is injective. Put $M := \coprod_{i=1}^r M_i$. The group $\prod_{i=1}^r \mathrm{Diff}^+(M_i)$ acts on M separately on each factor. Thus there is a smooth M-bundle $E \to B = \prod_{i=1}^r B\,\mathrm{Diff}^+(M_i)$. The diagram

$$\begin{tikzcd} W \arrow[r, "\kappa^{n,0}"] \arrow[d, "\kappa_E"] & H^*(\mathcal{B}_n^0) \arrow[d] \\ H^*(\prod_{i=1}^r B\,\mathrm{Diff}^+(M_i)) \arrow[r] & H^*(\coprod_{i=1}^r B\,\mathrm{Diff}^+(M_i)) \end{tikzcd}$$

(the bottom map comes from the natural map) commutes and therefore $\kappa_E : W \to H^*(B)$ is injective. The purpose of this argument is to show that we can find a single manifold M and a smooth M-bundle $f : E \to B$ on a connected base space such that $\kappa_E : W \to H^*(B)$ is injective.

Let $m \in \mathbb{N}$ and let Σ_m be the symmetric group. Now we consider

$$\begin{tikzcd} E' \arrow[r, "p'"] \arrow[d, "f'"] & E \arrow[d, "f"] \\ E(\Sigma_m; \underline{m} \times B^m) \arrow[r, "p"] \arrow[d, "q"] & B \\ E(\Sigma_m; B^m); & \end{tikzcd} \tag{6.2}$$

the map p is given by the Σ_m-equivariant map $\underline{m} \times \ni (i, x_1, \ldots, x_m) \mapsto x_i \in B$; the square is a pullback and the composition $q \circ f'$ is a smooth $\underline{m} \times M$-bundle (note the similarity to the loop space construction).

In the same way as in 4.4, one sees that the diagram

$$\begin{tikzcd} W \arrow[r, "\kappa_E"] \arrow[dr, "\kappa_{E'}"] & H^*(B) \arrow[d, "q_! \circ p^*"] \\ & H^*(E(\Sigma_m; B^m)) \end{tikzcd} \tag{6.3}$$

commutes. Hence the induced diagram

$$\begin{tikzcd} \Lambda W \arrow[r, "\Lambda\kappa_E"] \arrow[dr, "\Lambda\kappa_{E'}"] & \Lambda H^*(B) \arrow[d, "\Lambda(q_! \circ p^*)"] \\ & H^*(E(\Sigma_m; B^m)) \end{tikzcd} \tag{6.4}$$

commutes as well. The top horizontal map is injective by assumption.

The map $q_! \circ p^* : H^*(B;\mathbb{Q}) \to H^*(E(\Sigma_m; B^m);\mathbb{Q})$ induces an algebra map $\Lambda\tilde{H}^*(B) \to H^*(E(\Sigma_m; B^m);\mathbb{Q})$ which is an isomorphism up to degree $m/2$. This is a combination of the Barratt-Priddy-Quillen theorem and homological stability for symmetric groups (Nakaoka et alii). See [9] for details and references. □

It is obvious that it is necessary to consider nonconnected manifolds in the above proof of Theorem **??**. We do not know whether Theorem B remains true if $\Lambda\kappa^n$ is replaced by $\Lambda\kappa^{n,0}$. In the 2-dimensional case, the situation is different. All published proofs of Theorem 2.2 show that $\Lambda\kappa^{n,0}$ is injective. For the passage from $\kappa^{2,0}$ to $\Lambda\kappa^{2,0}$, the use of Harer's homological stability theorem for the mapping class groups is essential, while the stability result is not necessary to show that $\kappa^{2,0}$ is injective (this point is most obvious in Miller's proof [19]). Since a large portion of the proof of Theorem A relies on the 2-dimensional case, there are partial results for $\Lambda\kappa^{n,0}$, see e.g. [12] for a result in the 4-dimensional case.

7. The holomorphic case

In this section, we prove Theorem C, which is parallel to the proofs of A and B. So we sketch only the differences.

The proofs of 2.2 given by Miller and Morita show that Theorem C holds if $m = 1$. The inductive procedure works in the same way; the proof of 2.5 is easily adjusted and shows that only the classes of the form $\kappa_E(\mathrm{ch}_{2d})$, $2d \geq 2m$ cannot be detected on products.

If $q : E \to BU(m+1)$ is the universal $\mathbb{CP}^m$-bundle, then the class $q_!(\mathrm{ch}_{2d}(T_vE))$ is nonzero if $2d \geq 2m$ and $d - m \equiv 0 \pmod 2$ by Theorem 2.9. Of course, $BU(m)$ is not a complex manifolds; nevertheless it can be approximated by the Grassmann manifolds $\mathrm{Gr}_m(\mathbb{C}^r)$ of m-dimensional quotients of $\mathbb{C}^r$ for $r \gg m$, which is a projective variety. The tautological vector bundle on $\mathrm{Gr}_m(\mathbb{C}^r)$ is a holomorphic vector bundle and hence its projectivization is a holomorphic fibre bundle.

Thus we are left with showing that $\kappa^m_{\mathbb{C}}(\mathrm{ch}_{2d}) \neq 0$ if $2m \leq 2d$ and $d - m \equiv 1 \pmod 2$. The loop space construction as in section 4 does not make sense in the holomorphic realm. One could replace $\mathbb{S}^1$ by $\mathbb{CP}^1$ in the loop space construction and the space $\mathrm{map}(\mathbb{S}^1, BU(m+1))$ by the approximating space $\mathrm{hol}_k(\mathbb{CP}^1; \mathrm{Gr}_{m+1}(\mathbb{C}^r)$ and then use the fact (proven by Segal and Kirwan) that the space of holomorphic maps into a Grassmannian is a good homotopical approximation to the space of all maps, bu we prefer a more direct route. Let $T \to \mathbb{CP}^1$ and $L \to \mathbb{CP}^r$ be the tautological line bundles. Consider the 2-dimensional vector bundle $V = (\mathbb{C} \oplus T) \boxtimes L \to \mathbb{CP}^1 \times \mathbb{CP}^r$. Its total Chern class is $c(V) = (1 + 1 \times x)(1 + z \times 1 + 1 \times x)$, where $x \in H^2(\mathbb{CP}^r)$ and $z \in H^2(\mathbb{CP}^1)$ are the usual generators. Therefore the second Chern class is $u = 1 \times x^2 + z \times x$ and $u^l = 1 \times x^{2l} + lz \times x^{2l-1} \neq 0$ for $r \gg 2l$. Consider the composite bundle

$$\mathbb{P}(V \oplus \mathbb{C}^{m-2}) \xrightarrow{q} \mathbb{CP}^1 \times \mathbb{CP}^r \xrightarrow{\mathrm{pr}} \mathbb{CP}^r$$

with fibre $\mathbb{CP}^1 \times \mathbb{CP}^{m-1}$. A computation similar to the one in 4.2 (and using Proposition **??**, (4)) shows that

$$\mathrm{pr}_! \, q_!(\mathrm{ch}(T^{proj \circ q})) = \mathrm{pr}_! \, q_!(\mathrm{ch}(T^q)) + \mathrm{pr}_!(q_! q^* \, \mathrm{ch}(T^{proj})) = \mathrm{pr}_! \, q_!(\mathrm{ch}(T^q)).$$

By Theorem 5.4 and Lemma 5.1

$$\mathrm{pr}_! \, q_!(\mathrm{ch}(T^q)) = \mathrm{pr}_!(\sum_{l=0}^{\infty} a_l \, \mathrm{pr}_! \, u^l),$$

where a_l is the nonzero rational number from Theorem 5.4. But $\mathrm{pr}_!(u^l) = l x^{2l-1}$. This finishes the proof that $\kappa_E \mathbb{C}(\mathrm{ch}_{2d}) \neq 0$ for a certain m-dimensional bundle with $m-d$ odd.

To show the second half of Theorem 7, we replace the space $E\Sigma_m$ by the configuration space $C^m(\mathbb{C}^r)$ of m numbered points in $\mathbb{C}^r$ for sufficiently large r.

Appendix A. Gysin maps and the transfer

Here we give a brief recapitulation of Gysin maps for fibre bundles. The Gysin homomorphism of a smooth closed oriented manifold bundle is defined by means of the Leray-Serre spectral sequence, see e.g. [21], p. 147 ff. Let $E_r^{p,q}$ be the Leray-Serre spectral sequence. The Gysin map $f_!$ is defined as the composition

$$\text{(A.1)} \qquad f_! : H^{k+n}(E) \to E_\infty^{k,n} \subset E_2^{k,n} = H^k(B; \underline{H^n(M)}) \stackrel{\cap [M]}{\to} H^k(B).$$

The last map arises as follows. Because the bundle is oriented, the fundamental class $[M]$ of the fibre defines a homomorphism $\underline{H^n(M)} \to \mathbb{Z}$ of coefficient systems on B (the system $\mathbb{Z}$ is the constant one); it is always an epimorphism and it is an isomorphism if M is connected. One can replace $\mathbb{Z}$ of course by any other ground ring. Below there is a list of the main properties of the Gysin map. The proof can be found in [5], section 8.

Proposition A.2. *Let M be a closed oriented n-manifold and $f : E \to B$ be smooth oriented M-bundle.*

(1) Naturality: *If*

$$\begin{array}{ccc} E' & \xrightarrow{\hat{g}} & E \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow{g} & B \end{array}$$

is a pullback-square, then $f'_! \circ \hat{g}^* = g^* \circ f_!$.

(2) $H^*(B)$-linearity: *If $x \in H^*(E)$ and $y \in H^*(B)$, then $f_!((f^*y)x) = y \times f_!(x)$.*

(3) Normalization: *If M is an oriented n-manifold with fundamental class $[M] \in H_n(M)$ and $f : M \to *$ the constant map, then $f_!(x) = \langle x; [M] \rangle 1$ for all $x \in H^*(M)$.*

(4) Transitivity: *If N is another closed oriented manifold and $g : X \to E$ be a smooth oriented N-bundle, then $(f \circ g)_! = f_! \circ g_!$.*

The following properties are straightforward consequences of Proposition A.2.

Proposition A.3. *Let $f : E \to B$ be an oriented smooth n-manifold bundle.*

(1) *Then $f_!(xf^*(y)) = (-1)^{|x||y|} f_!(f^*(y)x) = (-1)^{(|x|-n)|y|} f_!(x)y$.*
(2) *Let $f_i : E_i \to B_i$, $i = 1, 2$, be two oriented fibre bundles of fibre dimension n_i. Consider the oriented fibre bundle $f = f_1 \times f_2 : E = E_1 \times E_2 \to B = B_1 \times B_2$ of fibre dimension $n = n_1 + n_2$. Then $(f_1 \times f_2)_!(x_1 \times x_2) = (-1)^{n_2|x|}(f_1)_!(x_1) \times (f_2)_!(x_2)$ for all $x_i \in H^*(E_i)$.*
(3) *If $f : E \to B$ is a homeomorphism (the fibre is a point), then $f_! = (f^{-1})^*$.*
(4) *If the fibres of f have positive dimension, then $f_! \circ f^* = 0$.*

Another construction of Gysin maps is homotopy-theoretic in nature and uses the Pontrjagin-Thom construction, see e.g. [4]. If $f : E \to B$ is a smooth manifold bundle with closed fibres, then the Pontrjagin-Thom map is a map $\mathrm{PT}_f : \Sigma^\infty B_+ \to \mathbb{T}\mathbf{h}(-T_v E)$ of spectra ($\mathbb{T}\mathbf{h}(-T_v E)$ is the Thom spectrum of the stable vector bundle $-T_v E$. The Thom isomorphism of $T_v E$ is an isomorphism $\mathrm{th} : H^*(\Sigma^\infty E_+) \cong H^{*-n}(\mathbb{T}\mathbf{h}(-T_v E))$ and the Gysin map is the composition

$$f_! = \mathrm{PT}_f^* \circ \mathrm{th} : H^k(E) = H^k(\Sigma^\infty E_+) \stackrel{\mathrm{th}}{\to} H^{k-n}(\mathbb{T}\mathbf{h}(-T_v E)) \stackrel{\mathrm{PT}_f^*}{\to} H^{k-n}(\Sigma^\infty B_+) = H^{n-k}(B).$$

Closely related to the Gysin map is the *transfer*.

Definition A.4. Let $f : E \to B$ be an oriented smooth bundle. Then the *transfer* is the map $\mathrm{trf}_f^* : H^*(E) \to H^*(B)$ given by $\mathrm{trf}_f^*(x) := f_!(\chi(T_v E)x)$.

Note that both $\chi(T_v E)$ and $f_!$ reverse their sign if the orientation of $T_v E$ is reversed, so trf_f^* does not depend on the orientation. In fact, trf_f^* is induced by a stable homotopy class $\mathrm{trf}_f : \Sigma^\infty B_+ \to \Sigma^\infty E_+$ which only depends on the bundle and not on the orientation; in fact this stable homotopy class can be defined for more general bundles than we consider here. We do not use this homotopy-theoretic perspective in this paper[1]. What we need to know are the following properties which are straightforward consequences of Proposition A.2.

Proposition A.5. *Let $f : E \to B$ be an oriented smooth n-manifold bundle.*

(1) *If $g : F \to E$ is another smooth oriented manifold bundle, then $\mathrm{trf}_{f \circ g}^* = \mathrm{trf}_f^* \circ \mathrm{trf}_g^*$.*
(2) *The composition $\mathrm{trf}_f^* f^* : H^*(B) \to H^*(B)$ is multiplication by the Euler number $\chi(M)$ of the fibre.*
(3) *If $f : E \to B$ is a homeomorphism (the fibre is a point), then $\mathrm{trf}_f^* = (f^{-1})^*$.*

[1]We use it implicitly in the proof of Theorem B, though.

References

[1] T. Akita; N. Kawazumi; T. Uemura: *Periodic surface automorphisms and algebraic independence of Morita-Mumford classes*, J. Pure Appl. Algebra 160 (2001), 1-11.

[2] M. F. Atiyah: *The signature of fibre bundles.* Global Analysis (Papers in Honor of K. Kodaira), 73-84. Univ. Tokyo Press, Tokyo (1969).

[3] M. F. Atiyah; R. Bott: *The moment map and equivariant cohomology.* Topology 23 (1984), 1-28.

[4] J. C. Becker, D. H. Gottlieb: *The transfer map and fiber bundles.* Topology 14 (1975), 1-12.

[5] A. Borel, F. Hirzebruch: *Characteristic classes and homogeneous spaces, I.* Am. J. Math 80 (1959), 458-538.

[6] R. O. Burdick: *Oriented manifolds fibered over the circle.* Proc. A.M.S. 17 (1966), 449-452.

[7] P. E. Conner: *The bordism class of a bundle space.* Mich. Math. J. 14 (1967), 289-303.

[8] J. Ebert: *A vanishing theorem for characteristic classes of odd-dimensional manifold bundles.* Preprint, arXiv:0902.4719 (2009).

[9] J. Ebert; J. Giansiracusa: *Pontrjagin-Thom maps and the homology of the moduli stack of stable curves.* Preprint, arXiv 0712.0702 (2007).

[10] S. Galatius; I. Madsen; U. Tillmann; M. Weiss: *The homotopy type of the cobordism category.* Acta Math. 202, (2009), 195-239.

[11] S. Galatius; O. Randal-Williams: *Momoids of moduli spaces of manifolds.* Preprint, arXiv:0905.2855 (2009).

[12] J. Giansiracusa: *The diffeomorphism group of a $K3$ surface and Nielsen realization.* J. London Math. Soc. 79 (2009), 701-718.

[13] F. Hirzebruch; T. Berger; R. Jung: *Manifolds and modular forms.* Aspects of Mathematics, E20. Friedr. Vieweg & Sohn, Braunschweig, (1992).

[14] K. Jänich: *On invariants with the Novikov additivity property.* Math. Ann. 184 (1969), 65-77.

[15] W. Lück; A. Ranicki: *Surgery obstruction of fibre bundles.* J. Pure Appl. Algebra 81 (1992), 139-189.

[16] I. Madsen; U. Tillmann: *The stable mapping class group and* $\mathbb{CP}^{\infty}_{-1}$. Invent. Math. 145 (2001), 509-544.

[17] I. Madsen; M. Weiss: *The stable moduli space of Riemann surfaces: Mumford's conjecture.* Ann. of Math. 165 (2007), 843-941.

[18] W. Meyer: *Die Signatur von Faserbündeln und lokalen Koeffizientensystemen.* Bonner Mathematische Schriften 53 (1972).

[19] E. Y. Miller: *The homology of the mapping class group.* J. Differential Geom. 24 (1986), 1-14.

[20] S. Morita: *Characteristic classes of surface bundles.* Invent. Math. 90 (1987), 551-577.

[21] S. Morita: *Geometry of characteristic classes.* Translations of Mathematical Monographs, 199. American Mathematical Society, Providence, (2001).

[22] D. Mumford: *Towards an enumerative geometry of the moduli space of curves.* Arithmetic and geometry, Vol. II, 271-328, Progr. Math., 36, Birkhäuser Boston, Boston, MA, 1983.

[23] W. D. Neumann: *Fibering over the circle within a bordism class.* Math. Ann. 192 (1971), 191-192.

[24] R. Sadykov: *Stable characteristic classes of smooth manifold bundles.* Preprint arXiv:0910.4770 (2009).

[25] R. Thom: *Quelques proprietées des varietées différentiables.* Comment. Math. Helv. 28, (1954) 17-86.

Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Bundesrepublik Deutschland

E-mail address: `ebert@math.uni-bonn.de`